Τύπος εφαπτόμενης ευθείας όταν η καμπύλη C δίνεται ως τομή δύο επιφανειών $F(x,y,z)=0$

$G(x,y,z)=0$

$$\boxed{\frac{x-x_0}{\left.\frac{\partial(F,G)}{\partial(y,z)}\right|_0} = \frac{y-y_0}{\left.\frac{\partial(F,G)}{\partial(z,x)}\right|_0} = \frac{z-z_0}{\left.\frac{\partial(F,G)}{\partial(x,y)}\right|_0}}$$ Ιακωβιανή

Τύπος κάθετου επιπέδου για καμπύλη C και $\vec{R} = u(t)\vec{i} + v(t)\vec{j} + w(t)\vec{k}$ διανυσματική που διέρχεται από το C. $(x-x_0)u'(t_0)+(y-y_0)v'(t_0)+(z-z_0)w'(t_0)=0$

ΠΑΡΑΔΕΙΓΜΑ 505

Θέμα 70 19/9/94

$I = \int_0^\infty e^{-x^2}dx$ ΛΥΣΗ $\Rightarrow I^2 = \int_0^\infty e^{-x^2}dx \int_0^\infty e^{-y^2}dy = \iint_0^\infty e^{-(x^2+y^2)}dx\,dy$ (1)

Θέτω $x = r\cos\theta$, $y = r\sin\theta$ $\left\{ \boxed{dx\,dy = \frac{\partial(x,y)}{\partial(r,\theta)}\,dr\,d\theta} \right.$ Ιακωβιανή όπου $\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r$

$dx\,dy = r\,dr\,d\theta$

(1) $\Rightarrow \int_0^{\pi/2}\int_0^\infty r e^{-r^2}\,dr\,d\theta = \frac{\pi}{4} \Rightarrow I = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$

$\boxed{\vec{F} = \vec{\nabla} f}$ ⇒ συνάρτηση δυναμικού f

$\boxed{\vec{\nabla}\times\vec{F} = 0}$ ⇒ συντηρητικό πεδίο

Κρίσιμα σημεία μιας $f(x,y,z) \Rightarrow \boxed{\vec{\nabla} f = 0} \Rightarrow \frac{\partial f}{\partial x} = 0,\ \frac{\partial f}{\partial y} = 0,\ \frac{\partial f}{\partial z} = 0$

Συνάρτηση δυναμικού f μιας $\vec{F} = (x+y+z,\ xyz,\ x+\frac{1}{2}yz)$ είναι $\vec{F} = \vec{\nabla} f \Rightarrow \left\{ \frac{\partial f}{\partial x} = x+y+z,\ \frac{\partial f}{\partial y} = xyz,\ \frac{\partial f}{\partial z} = x+\frac{1}{2}yz \right.$

1) $\int \frac{\partial f}{\partial x}dx = f = f(x,y,z) + c(y,z)$

2) $\frac{df}{dy} = \frac{\partial f}{\partial y}$

3) $\frac{df}{dz} = \frac{\partial f}{\partial z}$

4) $\boxed{f = \ldots}$

Παράγωγος της $f(x,y,z)$ κατά τη κατεύθυνση του $V=(2,-1,2)$ στο P

$\left.\frac{\partial f}{\partial \vec{V}}\right|_{P(1,1,2)} = \vec{\nabla} f|_P \cdot \vec{V}_0 = (1,4,1)\left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right) = \frac{2}{3} - \frac{4}{3} + \frac{2}{3} = 0$

$\vec{\nabla} f|_P = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (y^2,\ 2xy+z,\ y) = (1,4,1)$

$\vec{V}_0 = \frac{(2,-1,2)}{\sqrt{4+1+4}} = \left(\frac{2}{3}, \frac{-1}{3}, \frac{2}{3}\right)$

το κάθετο διάνυσμα στην F

$$\text{grad} f = \vec{\nabla} f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

$$\text{curl}\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Αν $\vec{\nabla}\times\vec{F} = 0$ τότε είναι συντηρητικό το διανυσματικό πεδίο

Εφαπτόμενο επίπεδο κ κάθετη γραμμή σε μια επιφάνεια

ΠΑΡΑΔΕΙΓΜΑ (για εφαπτόμενο επίπεδο κ' κάθετη γραμμή)

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} + \frac{(z-z_0)^2}{c^2} = 1$$

ΛΥΣΗ

για εφαπτόμενο επίπεδο

1) $F = \frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} + \frac{(z-z_0)^2}{c^2} - 1 = 0$

2) $\vec{\nabla}\vec{F}\Big|_{P(x_P, y_P, z_P)} = \frac{2(x_P-x_0)}{a^2}\vec{i} + \frac{2(y_P-y_0)}{b^2}\vec{j} + \frac{2(z_P-z_0)}{c^2}\vec{k}$

3) Η εξίσωση του εφαπτόμενου επιπέδου είναι $\boxed{(\vec{r}-\vec{r}_P)\,\vec{\nabla}F\big|_P = 0}$ σοσ $\Rightarrow$ εξσυνάρτηση $(x-x_P)\frac{\partial F}{\partial x}\Big|_P + (y-y_P)\frac{\partial F}{\partial y}\Big|_P + \ldots$

$$\Rightarrow (x-x_P)\left[\frac{(x_P-x_0)}{a^2}\right] + (y-y_P)\left[\frac{(y_P-y_0)}{b^2}\right] + (z-z_P)\left[\frac{(z_P-z_0)}{c^2}\right] = 0$$

για κάθετη γραμμή

1) Η εξίσωση της κάθετης ευθείας είναι: $(\vec{r}-\vec{r}_P)\times\vec{\nabla}F\big|_P = 0 \Rightarrow \boxed{\dfrac{x-x_P}{\frac{\partial F}{\partial x}\big|_P} = \dfrac{y-y_P}{\frac{\partial F}{\partial y}\big|_P} = \dfrac{z-z_P}{\frac{\partial F}{\partial z}\big|_P}}$ σοσ

εξσυνάρτηση

$$\frac{x-x_P}{\frac{2(x_P-x_0)}{a^2}} = \frac{y-y_P}{\frac{2(y_P-y_0)}{b^2}} = \frac{z-z_P}{\frac{2(z_P-z_0)}{c^2}}$$

Κάθετο επίπεδο και εφαπτόμενη ευθεία σε μια καμπύλη

ΠΑΡΑΔΕΙΓΜΑ

Να βρεθεί η εφαπτόμενη ευθεία στο $O(x_0, y_0)$ της $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

ΛΥΣΗ

1) $f = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ 2) $(\vec{r}-\vec{r}_0)\,\vec{\nabla}f\big|_0 = (x-x_0)\frac{\partial f}{\partial x}\Big|_0 + (y-y_0)\frac{\partial f}{\partial y}\Big|_0 = \left((x-x_0)\frac{2x_0}{a^2} + (y-y_0)\frac{2y_0}{b^2} = 0\right)$

Αν $\boxed{y' = \dfrac{\alpha_1 x + \beta_1 y + \gamma_1}{\alpha_2 x + \beta_2 y + \gamma_2}}$ και $\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} \neq 0$ τότε ... $\boxed{y = y_0 + v}$ ...

$y' = f\left(\frac{y}{x}\right)$

$\alpha_1 x + \beta_1 y + \gamma_1 = 0$ (1)
$\alpha_2 x + \beta_2 y + \gamma_2 = 0$

ΠΑΡΑΔΕΙΓΜΑ 1

$\boxed{y' = (x+y)^2}$, θέτω $x + y = u \Rightarrow y' = u' - 1 \Rightarrow u' - 1 = u^2 \Rightarrow u' = 1 + u^2 \Rightarrow \frac{u'}{1+u^2} = 1 \Rightarrow \int \frac{1}{1+u^2} du = \int dx$

$\Rightarrow \tan^{-1}(u) = x + C \Rightarrow u = \tan(x + C) \Rightarrow x + y = \tan(x+C) \to \boxed{y = \tan(x+C) - x}$

$y' + p(x) y = q(x)$

ΛΥΣΗ

$y(x) = \frac{1}{\mu(x)} \left[\int \mu(x) q(x)\, dx + C \right]$ όπου $\mu(x) = \exp\left[\int p(x)\, dx\right]$ εκφώνηση

$y' + p(x) y = q(x) y^n$ (Bernoulli)

ΛΥΣΗ

Θέτω $y = z^{1/(1-n)}$

$y = x f(y') + g(y')$ (Lagrange) π.χ. $y = x(1+y') + (y')^4$

ΛΥΣΗ

Θέτω $s = y'$, $\boxed{\frac{dx}{ds} - \frac{f'(s)}{s - f(s)} x = \frac{g'(s)}{s - f(s)}}$ εκφώνηση $\Rightarrow x'(s) + p(s) x(s) = g(s) \Rightarrow \boxed{x(s) = \frac{1}{\mu(s)} \left[\int \mu(s) g(s) + \ldots \right.}$

$\boxed{y(s) = x(s) f(s) + g(s)}$

$y' = f_0(x) + f_1(x) y + f_2(x) y^2$ με μία γνωστή μερική λύση (Riccati)

ΠΑΡΑΔΕΙΓΜΑ

$y' = x^2 + 1 - 2xy + y^2$

$y_1 = x$, $\boxed{z' + [f_1(x) + 2 f_2(x) y_1] z = -f_2(x)}$ εκφώνηση γενικά όπου $y = y_1 + \frac{1}{z}$

$\Rightarrow$ ~~...~~ $z'(x) = -1 \Rightarrow z(x) = -x + C \Rightarrow \ldots$

$P(x,y) dx + Q(x,y) dy = 0 \Leftrightarrow y' = f(x,y)$

ΛΥΣΗ

Αν $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow$ είναι πλήρης, άρα υπάρχει $\boxed{F(x,y) = \int P(x,y) dx + \int Q(x,y) dy - \int \left[\frac{\partial}{\partial y} \int P(x,y) dx \ldots \right.}$ εκφώνηση

$y''+2ay'+by=g(x)$ Μέθοδος προσδιοριστέων συντελεστών

ΛΥΣΗ

1) $y(x)=y_0(x)+y_p(x)$

Για την $y_0(x)$

2) $y''+2ay'+by=0$

Θα έχω την $r^2+2ar+b=0 < \begin{matrix} r_1=\dots \\ r_2=\dots \end{matrix}$, $\boxed{\begin{matrix} y_0(x)=C_1e^{r_1x}+C_2e^{r_2x} \text{ για πραγματικούς} \\ y_0(x)=e^{ax}(C_1\cos\beta x+C_2\sin\beta x) \text{ για μιγαδικούς όπου } r_{1,2}=a\pm i\beta \end{matrix}}$

3) Για την $y_p(x)$

$g(x)=e^{Ax}(P_1\cos Bx+P_2(x)\sin Bx)$

$A+Bi=\dots$ είναι/δεν είναι ρίζα του $r^2+2ar+b=0$, άρα $d=0$ (αν δεν είναι)

$y_p(x)=x^d e^{Ax}[C_1\cos Bx+C_2\sin Bx]$ d = αριθμός πολλαπλότητας της ρίζας

$\max(P_1,P_2)$

$y''+p(x)y'+q(x)y=g(x)$. Μέθοδος μεταβολής των παραμέτρων

ΛΥΣΗ

1) $y''+p(x)y'+q(x)y=0$. Προσπαθώ να βρω εμφανώς μία λύση $y_1(x)\neq0$. Τη δεύτερη θα τη βρω απ' το τύπο $\boxed{y_2(x)=y_1(x)\int\frac{1}{[y_1(x)]^2}\exp\left[-\int p(x)dx\right]dx}$ ← ισχύει

2) $W_{12}=y_1y_2'-y_1'y_2\neq0 \rightarrow$ οι $y_1(x), y_2(x)$ είναι γρ. ανεξάρτητες λύσεις

3) $y(x)=C_1y_1(x)+C_2y_2(x)-y_1(x)\int\frac{y_2(x)g(x)}{W_{12}(x)}dx+y_2(x)\int\frac{y_1(x)g(x)}{W_{12}(x)}dx$

ΕΞΕΤΑΣΕΙΣ (27-6-2003)

ΘΕΜΑ 2ο μέθοδος μεταβολής των παραμέτρων (σελ 151)

$$\boxed{y(x) = c_1 e^{-x} + c_2 e^{-2x} - e^{-x}(-x) - e^{-x}}$$

$$y'' + 3y' + 2y = e^{-x} \quad (1)$$

$$y(x) = c_1 e^{-x} + c_2 e^{-2x} - e^{-x}\int \frac{e^{-2x} e^{-x}}{-e^{-3x}}dx + e^{-2x}\int \frac{e^{-x} e^{-x}}{-e^{-3x}}dx$$

$r^2 + 3r + 2 = 0$, $\Delta = 9 - 4\cdot 2 = 1$, $r_{12} = \frac{-3 \pm 1}{2} < \begin{matrix} -2 \\ -1 \end{matrix}$, $\boxed{y_1(x) = e^{-2x}}$, $\boxed{y_2(x) = e^{-x}}$

Η ομογενής εξίσωση της εξίσωσης (1), που είναι $y'' + 3y' + 2y = 0$ έχει μερικές λύσεις τις $y_1 = e^{-x}$, $y_2 = e^{-2x}$ και η Βρονσκιανή είναι:

$W_{12} = y_1 y_2' - y_1' y_2 = e^{-x}(-2)e^{-2x} + e^{-x}e^{-2x} = -2e^{-3x} + e^{-3x} = -e^{-3x} \neq 0$ άρα είναι γρ. ανεξάρτητες λύσεις

Θέτω $y_p = C_1(x)e^{-2x} + C_2(x)e^{-x}$ όπου y_p μια μερική λύση της μη ομογενούς

Θέτω $C_1'(x)e^{-2x} + C_2'(x)e^{-x} = 0 \quad (2)$

$C_1'(x)e^{-2x}(-2) + C_2'(x)e^{-x}(-1) = e^{-x} \Rightarrow -2C_1'(x)e^{-2x} - C_2'(x)e^{-x} = e^{-x} \quad (3)$

$(3) \xRightarrow{(2)} -2C_1'(x)e^{-2x} + C_1'(x)e^{-2x} = e^{-x} \Rightarrow C_1'(x) = -\frac{e^{-x}}{e^{-2x}} \Rightarrow C_1'(x) = -e^{x}$

$(2) \Rightarrow C_2'(x)e^{-x} = -C_1'(x)e^{-2x} \Rightarrow C_2'(x) = \frac{+e^{x}e^{-2x}}{e^{-x}} = 1$

$C_1(x) = \int -e^{x}dx = -e^{x}$

$C_2(x) = \int dx = x$

Άρα $y_p = -e^{x}e^{-2x} + x\cdot e^{-x} = -e^{-x} + xe^{-x} = e^{-x}(x-1)$

Άρα η γενική λύση είναι $\boxed{y(x) = c_1 e^{-x} + c_2 e^{-2x} + e^{-x}(x-1)}$

ΘΕΜΑ 3ο ΘΕΩΡΗΜΑ σελ 118

$(1+x)y'' + xy' - y = 0 \Rightarrow y'' + \frac{x}{1+x}y' - \frac{1}{1+x}y = 0 \qquad p(x) = \frac{x}{1+x}, \quad q(x) = \frac{-1}{1+x}$

Έχω τη μια μερική λύση $y_1(x) = e^{ax}$, $a \in \mathbb{R}$ [1η μερική λύση]

Θα βρω μια δεύτερη μερική λύση: $y_2(x) = y_1(x)\int v(x)dx$

όπου $v(x) = \frac{1}{[y_1(x)]^2}\exp\left[-\int p(x)dx\right] = \frac{1}{e^{2ax}}\exp\left[-\int \frac{x}{1+x}dx\right] = \frac{1}{e^{2ax}}\exp\left[-\int\left(1 - \frac{1}{1+x}\right)dx\right] = \frac{1}{e^{2ax}}\exp\left[-x + \ln\right.$

$= \frac{1}{e^{2ax}}\frac{1}{e^{x}}|x+1| = \frac{|x+1|}{e^{(2a+1)x}}$

$$\begin{array}{r|l} x & x+1 \\ -x-1 & 1 \\ \hline -1 & \end{array} \qquad \boxed{x = x+1-1}$$

$y_2(x) = e^{ax}\int \frac{|x+1|}{e^{(2a+1)x}}dx = e^{ax}\int \frac{x}{e^{cx}}dx + \int \frac{1}{e^{cx}}dx =$

$W_{12} = y_1 y_2' - y_2 y_1' = \dots \neq 0$ άρα είναι οι y_1, y_2 γρ. ανεξάρτητες

Άρα η γενική λύση είναι $y(x) = C_1 e^{\alpha x} + C_2 (\quad)$, C_1, C_2 : αυθαίρετες σταθερές

$$3 = C_1 e^0 + C_2 (\quad) = C_1 + C_2 (\quad)$$

ΘΕΜΑ 4ο

$$\det\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} = 2 - 2 = 0$$ είναι $\alpha_1 \alpha_2 \beta_1 \beta_2 = 4 \neq 0$ άρα $\frac{\alpha_1}{\alpha_2} = \frac{\beta_1}{\beta_2} = 2$

Θέτω $x + y(x) = \omega(x) \Rightarrow \frac{dx}{dx} + \frac{dy}{dx} = \frac{d\omega}{dx} \Rightarrow \boxed{\frac{d\omega}{dx} - \frac{dy}{dx} = 1}$

$$\frac{dy}{dx} = \frac{2x+2y+1}{x+y+2} \Rightarrow \frac{d\omega}{dx} - 1 = \frac{2\omega+1}{\omega+2} \Rightarrow \frac{d\omega}{dx} = 1 + \frac{2\omega+1}{\omega+2} \Rightarrow \frac{d\omega}{dx} = \frac{3(\omega+1)}{\omega+2} \Rightarrow dx = \frac{d\omega(\omega+2)}{3(\omega+1)} \Rightarrow$$

$$\Rightarrow \int dx = \frac{1}{3}\int \frac{\omega+1+1}{\omega+1} d\omega \Rightarrow x + C = \frac{1}{3}\left[\omega + \ln|\omega+1|\right] \Rightarrow x + C = \frac{1}{3}(x+y) + \frac{1}{3}\ln|x+y+1|$$

Για $x = 0 \Rightarrow y = 1$ άρα $0 + C = \frac{1}{3}(0+1) + \frac{1}{3}\ln|0+1+1| \Rightarrow C = \frac{1}{3} + \frac{\ln 2}{3} = \frac{1+\ln 2}{3}$

Άρα $\boxed{\frac{2}{3}x + \frac{1+\ln 2}{3} = \frac{y}{3} + \frac{\ln|x+y+1|}{3}}$

ΕΞΕΤΑΣΕΙΣ (27/05/2003)

$\boxed{\text{ΘΕΜΑ 8}}$

$$\begin{cases} x = u + \frac{v}{2} \\ y = v \end{cases} \quad \begin{cases} dx = du + \frac{dv}{2} \\ dy = dv \end{cases} \Rightarrow dx\,dy = dv\left(du + \frac{dv}{2}\right) = du\,dv + \frac{dv^2}{2}$$

$$I = \int_0^2 \left(\int_{y/2}^{(y+4)/2} y^3 (2x-y) e^{(2x-y)^2} dx \right) dy$$

$$\int_{v/2}^{(v+4)/2} v^3 (2u + \cancel{v} - \cancel{v}) e^{(2u+\cancel{v}-\cancel{v})^2} \left(du + \frac{dv}{2}\right)$$

$$\int_{v/2}^{(v+4)/2} 2uv^3 e^{4u^2} du + \int_{v/2}^{(v+4)/2} uv^3 e^{4u^2} dv = 2v^3 \int_{v/2}^{(v+4)/2} u \cdot e^{4u^2} du + ue^{4u^2} \int_{v/2}^{(v+4)/2} v^3 dv =$$

$$= \frac{2v^3}{8} \left[e^{4u^2} \right]_{v/2}^{(v+4)/2} + \frac{ue^{4u^2}}{4} \left[v^4 \right]_{v/2}^{(v+4)/2} = \frac{v^3}{4} \left[e^{(v+4)^2} - e^{v^2} \right] + \frac{ue^{4u^2}}{4} \left[\frac{(v+4)^4}{16} - \frac{v^4}{16} \right] =$$

$$= \frac{v^3}{4} \left(e^{(v+4)^2} - e^{v^2} \right) + \frac{ue^{4u^2} \left((v+4)^4 - v^4 \right)}{64}$$

$$I = \int_0^2 \left[\frac{v^3}{4} \left(e^{(v+4)^2} - e^{v^2} \right) + \frac{ue^{4u^2} \left((v+4)^4 - v^4 \right)}{64} \right] dv$$

$$= \frac{1}{4} \int_0^2 v^3 \left(e^{(v+4)^2} - e^{v^2} \right) dv + \frac{ue^{4u^2}}{64} \int_0^2 (v+4)^4 - v^4 \, dv = \frac{1}{4} \Big[\qquad \Big]_0^2 + \frac{ue^{4u^2}}{64} \left[\frac{(v+4)^5}{5} - \frac{v^5}{5} \right]_0^2 (1)$$

$$\int_0^2 \left[e^{\ln v^3} e^{(v+4)^2} - e^{\ln v^3} e^{v^2} \right] = \int_0^2 \left(e^{3\ln v + (v+4)^2} - e^{3\ln v + v^2} \right) = \int_0^2 \left(e^{3\ln v + v^2 + 8v + 16} - e^{3\ln v + v^2} \right) dv = ;$$

$$(1) \Rightarrow \frac{1}{4} \Big[\qquad \Big]_0^2 + \frac{ue^{4u^2}}{64} \Big[\underbrace{\frac{6^5}{5} - \frac{2^5}{5}}_{\frac{6720}{5} = 1344} - \frac{4^5}{5} \Big]$$

$\boxed{\text{ΘΕΜΑ 1}}$ Μέθοδος των προσδιοριστέων συντελεστών

$y''-4y'+5y=8\cos x$ (1)

Έστω η ζητούμενη μερική λύση: $y=\alpha\cos x+\beta\sin x$

$[\alpha(-\sin x)]'+(\beta\cos x)'-4[\alpha(-\sin x)+\beta\cos x]+5\alpha\cos x+5\beta\sin x=8\cos x$

$-\alpha\cos x-\beta\sin x+4\alpha\sin x-4\beta\cos x=8\cos x$

$\cos x(-\alpha-4\beta-8+5\alpha)+\sin x(4\alpha-\beta+5\beta)=0$

$-8+4\alpha+4\beta=0 \Rightarrow -8+4\alpha+4\alpha=0 \Rightarrow 8\alpha-8=0 \Rightarrow \boxed{\alpha=1,\ \beta=-1}$

$4\alpha+4\beta=0 \Rightarrow \alpha=-\beta$

Άρα μια μερική λύση είναι $y_p=\cos x-\sin x$.

Η γενική λύση της $y''-4y'+5y=0$ βρίσκεται ως εξής:

$r^2-4r+5=0$, $\Delta=16-4\cdot 5=16-20=-4<0$, $r_{1,2}=\frac{4\pm i2}{2}$ $\begin{cases}2+i\\2-i\end{cases}$ $\boxed{\alpha=2}$ $\boxed{\beta=1}$

$y=e^{2x}[C_1\cos x+C_2\sin x]$

Άρα η γενική λύση της (1) είναι $y=e^{2x}[C_1\cos x+C_2\sin x]+\cos x-\sin x$

~~Θέματα~~ ΕΞΕΤΑΣΕΙΣ (27-6-03)

ΘΕΜΑ 6ο

$\vec{F} = (2xy^3 - ye^z, x(3xy^2 - e^z), -xye^z)$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^3 - ye^z & 3x^2y^2 - xe^z & -xye^z \end{vmatrix} = \dots = 0$$ Άρα $\vec{F}$ συντηρητικό πεδίο

ΘΕΜΑ 7ο

$f = f(x,y) = x^3 + y^3 + 3x^2 - 6y^2$

Για τα κρίσιμα σημεία της f πρέπει $\nabla f = 0 \Rightarrow \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = 0 \Rightarrow$

$$\Rightarrow \begin{cases} \frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2 + 6x = 0 \Rightarrow x^2 + 2x = 0 \Rightarrow x(x+2) = 0 \Rightarrow x = 0 \text{ ή } x = -2 \\ \frac{\partial f}{\partial y} = 3y^2 - 12y = 0 \Rightarrow y^2 - 4y = 0 \Rightarrow y(y-4) = 0 \Rightarrow y = 0 \text{ ή } y = -4 \end{cases}$$

$A(0,0)$, $B(-2,-4)$, $\Gamma(0,-4)$, $\Delta(-2,0)$

ΘΕΜΑ 5ο

$C_1: x^3 - 2xy^2 + 2y - z + 3 = 0 = F(x,y,z)$

$C_2: x^2 + 2y^2 + xy - z - 11 = 0 = G(x,y,z)$

$$\vec{\nabla} F = \frac{\partial F}{\partial x}\vec{i} + \frac{\partial F}{\partial y}\vec{j} + \frac{\partial F}{\partial z}\vec{k} = (3x^2 - 2y^2)i + (-4xy + 2)j + (-1)\vec{k}$$

$$\vec{\nabla} G = \frac{\partial G}{\partial x}\vec{i} + \frac{\partial G}{\partial y}j + \frac{\partial G}{\partial z}\vec{k} = (2x + y)i + (4y + x)j + (-1)\vec{k}$$

Στο $A(1,2,0)$ είναι $\vec{\nabla} F = -5i - 6j - k$ είναι το κάθετο διάνυσμα στην επιφάνεια C_1

$\vec{\nabla} G = 4i + 9j - k$ -||- -||- -||- C_2

~~Το~~ $\vec{T}_A = \vec{\nabla} F \times \vec{\nabla} G$ εφαπτόμενο της καμπύλης στο A

και η εξίσωση εφαπτομένης: $\frac{\partial(F,G)}{\partial(y,z)}(x - x_0) + \frac{\partial(F,G)}{\partial(z,x)}(y - y_0) + \frac{\partial(F,G)}{\partial(x,y)}$

ΘΕΜΑ 6ο

Συνάρτηση δυναμικού f:

$f: \boxed{\vec{F} = \nabla f}$

$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$

$\frac{\partial f}{\partial x} = 2xy^3 - ye^z \Rightarrow \boxed{f = x^2y^3 - ye^z x + c(y,z)}$ (1)

$\frac{\partial f}{\partial y} = 3x^2y^2 - xe^z \Rightarrow \cancel{3x^2y^2} - \cancel{xe^z} = \cancel{3x^2y^2} - \cancel{e^z x} + \frac{\partial c(y,z)}{\partial y} \Rightarrow \boxed{c(y,z) = c(z)}$ (2)

$\frac{\partial f}{\partial z} = -xye^z$ $\quad (1) \xRightarrow{(2)} f = (x^2y^3 - ye^z x + c(z)$

$\cancel{-xye^z} = \cancel{-yxe^z} + \frac{\partial c(z)}{\partial z} \Rightarrow \boxed{c(z) = c}$

$\boxed{f = x^2y^3 - ye^z x + c}$

I =

27/6/2003

ΘΕΜΑ 1ο

$$y'' - 4y' + 5y - 8\cos x = 0$$

$$\Rightarrow y'' - 4y' + 5y = 8\cos x$$

Είναι γραμμική μη ομογενής.
Άρα η λύση της είναι:

$$y(x) = y_0(x) + y_p(x)$$

• Για την $y_0(x)$ (η λύση της ομογενούς)

$$y'' - 4y' + 5y = 0$$

Η λύση είναι της μορφής $y_0(x) = e^{rx}$ για το r λύνω την εξίσωση:

$$r^2 - 4r + 5 = 0 \rightsquigarrow$$ χαρακτ. πολυώνυμο

$$\Delta = \beta^2 - 4\alpha\gamma = (-4)^2 - 4\cdot 5 = 16 - 20 = -4 = $$
$$= 4i^2$$

$$r_{1,2} = \overset{\frac{-\beta \pm \sqrt{\Delta}}{2\alpha}}{=} \frac{4 \pm 2i}{2} = \begin{cases} 2 + i = r_1 \\ 2 - i = r_2 \end{cases} \quad \kappa = 2,\ \lambda = 1$$

Άρα $y_0(x) = e^{\kappa x}\left[c_1 \cos\lambda x + c_2 \sin\lambda x\right]$

$$\Rightarrow y_0(x) = e^{2x}\left[c_1 \cos x + c_2 \sin x\right]$$

• Για την $y_{\mu}(x)$ (μερική λύση)

$$y''(x) - 4y'(x) + 5y(x) = \underbrace{8\cos x}_{f(x)}$$

$$f(x) = e^{Ax}\left[P_r(x)\cos Bx + P_m(x)\sin Bx\right]$$

$$8\cos x = e^{0x}\left[8\cos 1x + 0\cdot\sin 1x\right]$$

$A+Bi \Rightarrow 0+1\cdot i = i$ δεν είναι ρίζα του χαρακτηριστικού πολυωνύμου, άρα $\lambda=0$

Η μερική λύση έχει μορφή:

$$y_{\mu}(x) = x^{\lambda}\cdot e^{Ax}\left[Q_1(x)\cos Bx + Q_2(x)\sin Bx\right]$$

$$= x^0 e^{0x}\left[c_3\cos x + c_4\sin x\right]$$

$$y_{\mu}(x) = c_3\cos x + c_4\sin x$$

* Q_1, Q_2 πολυώνυμα βαθμού $= \max\{r, m\}$

Θέμα 2ο   Σελ. 151   Διαφορικές

$$y'' + 3y' + 2y = e^{-x}$$

$$y(x) = y_0(x) + y_u(x)$$

• Για την $y_0(x)$:

$$y'' + 3y' + 2y = 0$$

$$r^2 + 3r + 2 = 0$$

$$\Delta = 3^2 - 2\cdot 4 = 9 - 8 = 1$$

$$r_{1,2} = \frac{-3 \pm \sqrt{1}}{2} = \frac{-3 \pm 1}{2} \begin{cases} \frac{-3+1}{2} = \frac{-2}{2} = -1 \\ \frac{-3-1}{2} = -\frac{4}{2} = -2 \end{cases}$$

$$y_0(x) = c_1 e^{-x} + c_2 e^{-2x}$$

• Άρα για τη $y_u(x)$:

$$y_u(x) = c_1(x)\, e^{-x} + c_2(x)\, e^{-2x}$$

$\Longrightarrow$

Για τα $C_1(x)$, $C_2(x)$ λύνω το σύστημα:

$$\begin{cases} C_1'(x) \cdot e^{-x} + C_2'(x)\, e^{-2x} = 0 \\ C_1'(x) \cdot (e^{-x})' + C_2'(x)\,(e^{-2x})' = e^{-x} \end{cases}$$

$$\Rightarrow \begin{cases} C_1'(x)\, e^{-x} + C_2'(x)\, e^{-2x} = 0 \\ -C_1'(x)\, e^{-x} - 2\, C_2'(x)\, e^{-2x} = e^{-x} \end{cases}$$

$$\Rightarrow \begin{cases} C_1'(x) + C_2'(x)\, e^{-x} = 0 \\ -C_1'(x) - 2\, C_2'(x)\, e^{-x} = 1 \end{cases}$$

Προσθέτω κατά μέλη:

$$-C_2'(x)\, e^{-x} = 1$$

$$\Rightarrow C_2'(x) = -e^{x}$$

$$\Rightarrow C_2(x) = -\int e^{x}\, dx$$

$$\Rightarrow \underline{C_2(x) = -e^{x}}$$

Αντικαθιστώ στην πρώτη και έχω:

$$C_1'(x) - e^{x} \cdot e^{-x} = 0$$

$$\Rightarrow C_1'(x) - 1 = 0$$

$$\Rightarrow C_1'(x) = 1$$

$$\Rightarrow \underline{C_1(x) = \int dx = x}$$

∴ Sea:

$$y_p(x) = x\,e^{-x} - e^{x}\,e^{-2x}$$
$$= x\,e^{-x} - e^{-x}$$
$$= \underline{(x-1)\,e^{-x}}$$

Sea $y(x) = y_0(x) + y_p(x)$

$$= \underline{c_1 e^{-x} + c_2 e^{-2x} + (x-1)\,e^{-x}}$$

ΘΕΜΑ 3º   Σελ. 118

$$(1+x)\,y'' + x\,y' - y = 0$$

Θέμα 4ο

$y(0)=1$

$$y' = \frac{2x+2y+1}{x+y+2} \quad D = \begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} = 2-2=0$$

$$\Rightarrow y' = \frac{2(x+y)+1}{(x+y)+2} \Bigg\} \; y' = \frac{2z+1}{z+2}$$

Θέτω $z = x+y$

$$\frac{dz}{dx} = 1 + \frac{dy}{dx}$$

$$z' = 1 + y' \quad \Rightarrow$$

$$\Rightarrow z' = 1 + \frac{2z+1}{z+2}$$

$$\Rightarrow \frac{dz}{dx} = \frac{z+2+2z+1}{z+2}$$

$$\Rightarrow \frac{dz}{dx} = \frac{3z+3}{z+2}$$

$$\Rightarrow dz\,(z+2) = 3(z+1)\,dx$$

$$\Rightarrow \frac{z+2}{3(z+1)}\,dz = dx$$

$$\Rightarrow \frac{1}{3}\int \frac{z+2}{z+1}\,dz = \int dx + c$$

$$\Rightarrow \frac{1}{3}\int \left(\frac{z+1}{z+1} + \frac{1}{z+1}\right) dz = x + c$$

$$\Rightarrow \frac{1}{3}\int \left(1 + \frac{1}{z+1}\right) dz = x + c$$

$$\Rightarrow \frac{1}{3}\int dz + \frac{1}{3}\int \frac{1}{z+1}\,dz = x + c$$

$$\Rightarrow \frac{1}{3} z + \frac{1}{3}\ln(z+1) = x + c$$

$$\left.\begin{array}{l} \Rightarrow \frac{1}{3}(x+y) + \frac{1}{3}\ln(x+y+1) = x + c \\ \\ y(0) = 1 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \frac{1}{3}(0+1) + \frac{1}{3}\ln(0+1+1) = 0 + c$$

$$\Rightarrow \underline{\frac{1}{3} + \frac{1}{3}\ln 2 = c}$$

Άρα η λύση είναι:

$$\frac{1}{3}(x+y) + \frac{1}{3}\ln(x+y+1) = x + \frac{1}{3} + \frac{1}{3}\ln 2$$

ΘΕΜΑ 8º

$$I=\int_{y=0}^{2}\int_{x=y/2}^{(y+4)/2} y^3(2x-y)\,e^{(2x-y)^2}\,dx\,dy$$

Θέτω $x = u + \frac{v}{2}$

$y = v$

$dx = du$

$dy = du + \frac{dv}{2}$

$dy = dv$ $\left(du + \frac{dv}{2}\right)$

$dy = dv$ $\Rightarrow$

$x = \frac{y}{2} \Rightarrow u + \cancel{\frac{v}{2}} = \cancel{\frac{v}{2}} \Rightarrow \boxed{u=0}$

$x = \frac{(y+4)}{2} \Rightarrow u + \frac{v}{2} = \frac{v+4}{2}$

$\Rightarrow u + \cancel{\frac{v}{2}} = \cancel{\frac{v}{2}} + 2$

$\Rightarrow \boxed{u = 2}$

$$\Rightarrow I = \int_{v=0}^{2}\int_{u=0}^{2} v^3\left[2\left(u+\frac{v}{2}\right)-v\right] e^{\left[2\left(u+\frac{v}{2}\right)-v\right]^2} du\,dv$$

$$= \int_{v=0}^{2}\int_{u=0}^{2} v^3\, 2u\, e^{4u^2}\, du\, dv$$

$$= \int_{v=0}^{2} v^3 \left(\int_{u=0}^{2} 2u\, e^{4u^2}\, du\right) dv$$

$$= \int_{v=0}^{2} v^3\, \frac{1}{4}\left(\int_{u=0}^{2} 8u\, e^{4u^2}\, du\right) dv$$

$$= \int_{v=0}^{2} \frac{v^3}{4}\left(\int_{u=0}^{2} \left(e^{4u^2}\right)' du\right) dv$$

$$= \int_{v=0}^{2} \frac{v^3}{4}\left(e^{4u^2}\right)_0^2 dv = \left(e^{16}-1\right)\int_{v=0}^{2} \frac{v^3}{4}\, dv =$$

$$= \frac{\left(e^{16}-1\right)}{4}\left(\frac{v^4}{4}\right)_0^2 = \underline{\frac{\left(e^{16}-1\right)}{4}\cdot 16}$$

ΘEMA 6º

$$\vec{F} = \left(2xy^3 - ye^z,\ x(3xy^2 - e^z),\ -xye^z\right)$$

$$= \left(2xy^3 - ye^z,\ 3x^2y^2 - xe^z,\ -xye^z\right)$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^3 - ye^z & 3x^2y^2 - xe^z & -xye^z \end{vmatrix}$$

$$= \vec{i} \cdot \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2y^2 - xe^z & -xye^z \end{vmatrix} + \vec{j} \cdot \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ 2xy^3 - ye^z & -xye^z \end{vmatrix} +$$

$$+ \vec{k} \cdot \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 2xy^3 - ye^z & 3x^2y^2 - xe^z \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}(-xye^z) - \frac{\partial}{\partial z}(3x^2y^2 - xe^z)\right) -$$

$$- \vec{j}\left(\frac{\partial}{\partial x}(-xye^z) - \frac{\partial}{\partial z}\left(2xy^3 - ye^z\right)\right)$$

$$= + \vec{k}\left(\frac{\partial}{\partial x}(3x^2y^2 - xe^z) - \frac{\partial}{\partial y}(2xy^3 - ye^z)\right)$$

$$= \vec{i}\left(-xe^{z} + xe^{z}\right) + \vec{j}\left(-ye^{z} + ye^{z}\right) +$$

$$+ \vec{k}\left(6xy^{2} - e^{z} - 6xy^{2} + e^{z}\right)$$

$$= \vec{i} \cdot 0 + \vec{j} \cdot 0 + \vec{k} \cdot 0 = \vec{0}$$

Άρα $\vec{F}$ συντηρητικό πεδίο.

ΘΕΜΑ 7ο

$$f = f(x,y) = x^3 + y^3 + 3x^2 - 6y^2$$

Για τα κρίσιμα σημεία της f πρέπει

$$\nabla f = 0 \Rightarrow \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \vec{0}$$

$$\Rightarrow \begin{cases} \frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2 + 6x = 0 \\ \frac{\partial f}{\partial y} = 0 \Rightarrow 3y^2 - 12y = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 3x(x+2) = 0 \\ 3y(y-4) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x = 0, \quad x = -2 \\ y = 0, \quad y = 4 \end{cases}$$

$A(0,0)$ $\quad B(-2,4)$

$\Gamma(0,4)$ $\quad \Delta(-2,0)$

[illegible]

[illegible]

[illegible]

[illegible]

7/2/2003

ΘΕΜΑ 1º

$$y^2 dx = (x^2 + 2xy)\, dy$$
$$\Rightarrow y^2 dx - (x^2 + 2xy)\, dy = 0 \quad (*)$$

$P(x,y) = y^2$
$Q(x,y) = -(x^2 + 2xy)$

$P(kx,ky) = (ky)^2 = k^2 y^2 = k^2 P(x,y)$
$Q(kx,ky) = -(k^2x^2 + 2k^2xy) =$
$= -k^2(x^2 + 2xy) =$
$= k^2 Q(x,y)$

Άρα η δοθείσα Δ.Ε. $(*)$ είναι ομογενής βαθμού ομογένειας $\lambda = 2$

$(*)$: $y^2 dx - (x^2 + 2xy)\, dy = 0$

Θέτω $u = \frac{y}{x} \Rightarrow y = ux$ $\quad dy = udx + xdu$

$dy = u\,dx + x\,du$ $\Rightarrow$

$\Rightarrow (ux)^2 dx - (x^2 + 2xux)(u dx + x du) = 0$
$\Rightarrow u^2x^2 dx - (x^2 + 2x^2u)(u dx + x du) = 0$
$\Rightarrow u^2\cancel{x^2} dx - \cancel{x^2}(1 + 2u)(u dx + x du) = 0$
$\Rightarrow u^2 dx - u(1 + 2u)\, dx - x(1 + 2u)\, du =$
$\Rightarrow (u^2 - u - 2u^2)\, dx - x(1 + 2u)\, du = 0$
$\Rightarrow (-u^2 - u)\, dx - x(1 + 2u)\, du = 0$

$$\Rightarrow \; -(u^2+u)\,dx - x\,(1+2u)\,du = 0$$

$$\Rightarrow \; (u^2+u)\,dx + x\,(1+2u)\,du = 0$$

$$\Rightarrow \; \frac{1}{x}\,dx + \frac{1+2u}{u^2+u}\,du = 0$$

$$\Rightarrow \; \int \frac{1}{x}\,dx + \int \frac{1+2u}{u^2+u}\,du = C$$

$$\Rightarrow \; \ln x + \ln(u^2+u) = C$$

$$\Rightarrow \; \ln\left[x\cdot(u^2+u)\right] = C$$

$$\Rightarrow \; x\cdot(u^2+u) = C_1 \quad (e^c)$$

$$\Rightarrow \; x\left(\frac{y^2}{x^2} + \frac{y}{x}\right) = C_1$$

$$\Rightarrow \; \frac{y^2}{x} + y = C_1$$

$\Rightarrow$ $A(-2, 1)$ άρα:

Για $x = -2$: $y = 1$:

$$\frac{1^2}{(-2)} + 1 = C_1$$

$$\Rightarrow \; -\frac{1}{2} + 1 = C_1 \;\Rightarrow\; C_1 = \frac{1}{2}$$

Άρα $\quad \dfrac{y^2}{x} + y = \dfrac{1}{2}$

ΘΕΜΑ 2ο

i) $$x^2 y'' - 2xy' + 2y = 0$$

Είναι διαφορική εξίσωση Euler

Θέτω $x = e^t \rightsquigarrow t = \ln x$

Άρα $dx = e^t dt$

$$\Rightarrow \frac{dt}{dx} = \frac{1}{e^t} = \frac{1}{x}$$

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$y'' = \frac{d^2y}{dx^2} = \left(\frac{1}{x}\frac{dy}{dt}\right)' = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2}\frac{dt}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}$$

Αντικαθιστώ στη δοθείσα:

$$x^2\left(-\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}\right) - 2x\frac{1}{x}\frac{dy}{dt} + 2y = 0$$

$$\Rightarrow -\frac{dy}{dt} + \frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = 0$$

$$\Leftrightarrow y''(t) - 3y'(t) + 2y(t) = 0$$

$\hookrightarrow$ Είναι γραμμική ομογενής με σταθερούς συντελεστές.

$$r^2 - 3r + 2 = 0$$

$$\Delta = (-3)^2 - 4 \cdot 2 = 9 - 8 = 1$$

$$r_{1,2} = \frac{3 \pm 1}{2} = \begin{cases} \frac{4}{2} = 2 \\ \frac{2}{2} = 1 \end{cases}$$

Άρα $y(t) = c_1 e^{2t} + c_2 e^{t}$

$\Rightarrow y(x) = c_1 e^{2\ln x} + c_2 e^{\ln x}$

$\Rightarrow y(x) = c_1 x^2 + c_2 x$

ii) $x^2 y'' - 2xy' + 2y = 2x^4$

$$y(x) = \underbrace{y_0(x)}_{\text{λύση της ομογενούς}} + \underbrace{y_\mu(x)}_{\text{μερική λύση}}$$

• Από το (i): $y_0(x) = c_1 x^2 + c_2 x$

• ~~Για την~~ $\cancel{y_\mu(x)}$:

$\cancel{f(x) = 2x^4 = e^{0x}\left[2x^4 \cos 0x + 0 \sin 0x\right]}$

$\cancel{A + Bi = 0 + 0i = 0}$ ~~δεν είναι ρίζα του χαρακτ. πολυωνύμου άρα~~ $\cancel{t =}$

• Για την $y_\mu(x)$:

$$y_\mu(x) = c_1(x)\cdot x^2 + c_2(x)\cdot x$$

Λύνω το σύστημα:

$$\begin{cases} c_1'(x)\cdot x^2 + c_2'(x)\cdot x = 0 \\ c_1'(x)\cdot 2x + c_2'(x) = 2x^4 \end{cases}$$

$$D = \begin{vmatrix} x^2 & x \\ 2x & 1 \end{vmatrix} = x^2 - 2x^2 = -x^2$$

$$D_1 = \begin{vmatrix} 0 & x \\ 2x^4 & 1 \end{vmatrix} = -2x^5$$

$$D_2 = \begin{vmatrix} x^2 & 0 \\ 2x & 2x^4 \end{vmatrix} = 2x^6$$

$$c_1'(x) = \frac{D_1}{D} = \frac{-2x^5}{-x^2} = 2x^3$$

$$\Rightarrow c_1'(x) = 2x^3$$

$$\Rightarrow c_1(x) = \int 2x^3\,dx$$

$$\Rightarrow c_1(x) = 2\frac{x^4}{4} \Rightarrow c_1(x) = \frac{x^4}{2}$$

$$C_2'(x) = \frac{D_2}{D} = \frac{2x^6}{-x^2} = -2x^4$$

$$\Rightarrow C_2(x) = \int -2x^4 \, dx = -2\frac{x^5}{5}$$

$$\Rightarrow C_2(x) = -\frac{2x^5}{5}$$

$$y_p(x) = \frac{x^4}{2} \cdot x^2 - \frac{2x^5}{5} \cdot x$$

$$= \frac{x^6}{2} - \frac{2x^6}{5}$$

$$= \frac{5x^6 - 4x^6}{10} = \frac{x^6}{10}$$

Άρα $y(x) = C_1 x^2 + C_2 x + \frac{x^6}{10}$

ΘΕΜΑ 3ο

$$y'' + 3y' - 4y = 25e^{x} + 16x$$

Η λύση είναι:

(*) $y(x) = y_o(x) + y_\mu(x)$

λύση της ομογενούς — μερική λύση

• Για την $y_o(x)$:

$$y'' + 3y' - 4y = 0$$

$$r^2 + 3r - 4 = 0$$

$$\Delta = 9 - 4\cdot(-4) = 9 + 16 = 25$$

$$r_{1,2} = \frac{-3 \pm 5}{2} = 1, \; -4$$

Άρα

$$y_o(x) = c_1 e^{x} + c_2 e^{-4x}$$

• Για τη $y_\mu(x)$:

$$y_\mu(x) = y_1(x) + y_2(x)$$

$y_1(x)$ λύση της $y'' + 3y' - 4y = 25e^x$

$y_2(x)$ λύση της $y'' + 3y' - 4y = 16x$ (*)

i) Για την $y_1(x)$:

$$25e^x = e^x\left[25\cos 0x + 0\sin 0x\right]$$

$A + Bi = 1 + 0i = 1$ $\quad r^2+3r-4=0 \Rightarrow 1+3-4=0$

ρίζα του χαρακτ. πολυωνύμου με πολλαπλότητα $\underline{\lambda = 1}$

$$y_1(x) = e^x \cdot x\left[C_3 \cancelto{1}{\cos 0x} + \cancelto{0}{C_4 \sin 0x}\right]$$

$$\Rightarrow y_1(x) = \underline{C_3 \cdot x \cdot e^x}$$

ii) Για τη $y_2(x)$:

$$16x = e^{0x}\left[16x\cos 0x + \cancel{0\sin 0x}\right]$$

$A+Bi = 0+0i = 0$ δεν είναι ρίζα του χαρακτ. πολυωνύμου άρα $\lambda=0$

$$y_2(x) = e^{0x}\cdot x^0\left[(ax+\beta)\cos 0x + \cancel{Q\sin 0x}\right]$$

$$y_2(x) = ax+\beta$$

$$y_2'(x) = a$$

$$y_2''(x) = 0$$

αντικαθιστώ στην (*)

Άρα $0+3a-4(ax+\beta) = 16x$

$\Rightarrow 3a-4ax-4\beta = 16x$

$\Rightarrow -4ax+3a-4\beta = 16x$

$$\Rightarrow \begin{cases} -4a = 16 \\ 3a-4\beta = 0 \end{cases} \Rightarrow \begin{cases} a=-4 \\ \beta=-3 \end{cases}$$

Άρα $y_2(x) = -4x-3$

Επομένως

$$y_1(x) = c_3 x e^{x} - 4x - 3$$

και

$$y(x) = c_1 e^{x} + c_2 e^{-4x} + c_3 x e^{x} - 4x - 3$$

ΘΕΜΑ 4º

$$y = (2+y')x + (y')^2$$

Είναι διαφορική εξίσωση Lagrange

Αν θέσω $q = y'$ έχω:

$$y = \underbrace{(2+q)}_{f(q)} x + \underbrace{q^2}_{g(q)}$$

Το γενικό ολοκλήρωμα αυτής είναι:
σε παραμετρική μορφή:

$$\begin{cases} x = x(c,q) \\ y = x(c,q) \cdot f(p) + g(q) \\ \quad = x(c,q) \cdot (2+q) + p^2 \end{cases}$$

Ιδιάζουσες λύσεις:

$$f(q) - q = 0$$
$$\Rightarrow 2 + \cancel{q} - \cancel{q} = 0 \Rightarrow 2 = 0 \quad \text{Άτοπο}$$

Άρα δεν έχει ιδιάζουσες λύσεις.

$$y = (2+p)x + p^2$$

$$\frac{dy}{dx} = \frac{d}{}$$

ΘΕΜΑ 8ο

$$I = \iint (1 - x^2 - y^2)^{1/2}\, dx\, dy$$

$$x^2 + y^2 = x$$

$\Rightarrow$

$$\Rightarrow I = \iint \left(1 - (x^2 + y^2)\right)^{1/2} dx\, dy$$

$$= \iint (1 - x)^{1/2}\, dx\, dy$$

$$= \int \left(\int (1-x)^{1/2} dx \right) dy$$

Θέτω $1 - x = t$ ~~$\Rightarrow x = 1-t$~~

$-dx = dt \Rightarrow dx = -dt$

$$= \int \left(-\int t^{1/2} dt \right) dy$$

$$= \int -\frac{2t^{3/2}}{3}\, dy = -\frac{2}{3}(1-x)^{3/2} y + C$$

25/10/2002

ΘΕΜΑ 1º

i) $x^2y' - x^2y^2 + 5xy - 3 = 0$

Εξετάζω αν η $y = \frac{3}{x}$ είναι λύση

$x^2\left(\frac{3}{x}\right)' - x^2\left(\frac{3}{x}\right)^2 + 5\cancel{x}\left(\frac{3}{\cancel{x}}\right) - 3 =$

$= \cancel{x^2}\left(-\frac{3}{\cancel{x^2}}\right) - \cancel{x^2}\cdot\frac{9}{\cancel{x^2}} + 5\cdot 3 - 3$

$= -3 - 9 + 15 - 3 =$

$= -12 + 12 = 0$

Άρα είναι λύση

ii) $x^2y' - x^2y^2 + 5xy - 3 = 0$

$\Rightarrow x^2y' = x^2y^2 - 5xy + 3$

$\Rightarrow y' = y^2 - \frac{5}{x}y + \frac{3}{x^2}$ (*)

Διαφορική εξίσωση Riccatti

Γνωρίζουμε μια ειδική της λύση την $y_1(x) = \frac{3}{x}$

Κάνω τον μετασχηματισμό

$$y(x) = y_1(x) + \frac{1}{u(x)}$$

$$\Rightarrow y(x) = \frac{3}{x} + \frac{1}{u}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{3}{x^2} - \frac{u'}{u^2}$$

Άρα η (*) γίνεται

$$-\frac{3}{x^2} - \frac{u'}{u^2} = \left(\frac{3}{x} + \frac{1}{u}\right)^2 - \frac{5}{x}\left(\frac{3}{x} + \frac{1}{u}\right) + \frac{3}{x^2}$$

$$\Rightarrow \cancel{-\frac{3}{x^2}} - \frac{u'}{u^2} = \cancel{\frac{9}{x^2}} + \frac{1}{u^2} + \frac{6}{x \cdot u} - \cancel{\frac{15}{x^2}} - \frac{5}{xu} + \cancel{\frac{3}{x^2}}$$

$$\Rightarrow -\frac{u'}{u^2} = \frac{1}{u^2} + \frac{1}{xu}$$

$$\Rightarrow -u' \cdot x = x + u$$

$$\Rightarrow x \cdot u' + u = -x$$

$$\Rightarrow u' + \underbrace{\frac{1}{x}}_{P(x)} u = \underbrace{-1}_{q(x)} \quad \leadsto \text{γραμμική}$$

$$\mu(x) = e^{\int P(x)\,dx} = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

Η λύση είναι:

$$u(x) = e^{-\int P(x)dx}\left[c + \int q(x)\, e^{\int P(x)dx} d\right.$$

$$\Rightarrow u(x) = e^{-\int \frac{1}{x}dx}\left[c + \int -1 \cdot e^{\int \frac{1}{x}dx} dx\right]$$

$$\Rightarrow u(x) = e^{-\ln x}\left[c - \int e^{\ln x} dx\right]$$

$$\Rightarrow u(x) = x^{-1}\left[c - \int x\, dx\right]$$

$$\Rightarrow u(x) = \frac{1}{x}\left[c - \frac{x^2}{2}\right]$$

$$\Rightarrow y(x) - \frac{3}{x} = \frac{1}{x}\left[c - \frac{x^2}{2}\right]$$

$$\Rightarrow y(x) = \frac{3}{x} + \frac{1}{x}\left[c - \frac{x^2}{2}\right]$$

ΘΕΜΑ 2º

$$y'' - 3y' + 2y = 3x e^{x}$$

$$y(x) = y_0(x) + y_{\mu}(x)$$

• Για την $y_0(x)$:

$$y'' - 3y' + 2y = 0$$

$$r^2 - 3r + 2 = 0$$

$$\Delta = (-3)^2 - 4 \cdot 2 = 9 - 8 = 1$$

$$r_{1,2} = \frac{3 \pm 1}{2} = 2, \ 2$$

$$y_0(x) = c_1 x^2 + c_2 x$$

• Για την $y_\varepsilon(x)$:

$$y_\varepsilon(x) = C_1(x)\,x^2 + C_2(x)\,x$$

$$\begin{cases} C_1'(x)\,x^2 + C_2'(x)\,x = 0 \\ C_1'(x)\,2x + C_2'(x) = 3xe^x \end{cases}$$

$$D = \begin{vmatrix} x^2 & x \\ 2x & 1 \end{vmatrix} = x^2 - 2x^2 = -x^2$$

$$D_1 = \begin{vmatrix} 0 & x \\ 3xe^x & 1 \end{vmatrix} = -3x^2e^x$$

$$D_2 = \begin{vmatrix} x^2 & 0 \\ 2x & 3xe^x \end{vmatrix} = 3x^3e^x$$

$$C_1'(x) = \frac{D_1}{D} = \frac{\cancel{-}3\cancel{x^2}e^x}{\cancel{-x^2}} = 3e^x$$

$$\Rightarrow C_1(x) = \int 3e^x \, dx$$

$$\Rightarrow \underline{C_1(x) = 3e^x}$$

$$C_2'(x) = \frac{D_2}{D} = \frac{3x^{\cancel{2}}e^x}{\cancel{-x^2}} = -3xe^x$$

$$\Rightarrow C_2(x) = -\int 3xe^x \, dx$$

$$\Rightarrow C_2(x) = -3xe^x + 3\int e^x dx$$

$$\Rightarrow C_2(x) = -3xe^x + 3e^x$$

$$\Rightarrow \underline{C_2(x) = 3e^x(1-x)}$$

Άρα $y_\mu(x) = 3e^x \cdot x^2 + 3e^x(1-x)x$

και

$$\underline{y(x) = C_1x^2 + C_2x + 3e^x x^2 + 3e^x(1-x)x}$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

ΘΕΜΑ 4º

$$y'' - 2y' = x^2$$

$$\Rightarrow y(x) = y_0(x) + y_\mu(x)$$

· Για τη $y_0(x)$:

$$\Rightarrow y'' - 2y' = 0$$

$$\Rightarrow r^2 - 2r = 0$$

$$\Rightarrow r(r-2) = 0 \quad \begin{cases} r_1 = 0 \\ r_2 = 2 \end{cases}$$

Άρα $y_0(x) = C_1 e^{0x} + C_2 e^{2x}$

$$\Rightarrow y_0(x) = C_1 + C_2 e^{2x}$$

· Για τη $y_\mu(x)$:

$$y_\mu(x) = C_1(x) + C_2(x) e^{2x}$$

$$\begin{cases} C_1'(x) + C_2'(x) e^{2x} = 0 \\ 0 + 2C_2'(x) e^{2x} = x^2 \end{cases}$$

$$C_2'(x) = \frac{1}{2} x^2 e^{-2x}$$

$$\Rightarrow C_2(x) = \frac{1}{2} \int x^2 e^{-2x}\, dx$$

$$\Rightarrow C_2(x) = -\frac{1}{4} \int x^2 (e^{-2x})'\, dx$$

$$\Rightarrow C_2(x) = -\frac{1}{4} x^2 e^{-2x} + \frac{1}{\cancel{4}_2} \int \cancel{2} x\, e^{-2x}\, dx$$

$$\Rightarrow C_2(x) = -\frac{1}{4} x^2 e^{-2x} + \frac{1}{2} \int x\, e^{-2x}\, dx$$

$$\Rightarrow C_2(x) = -\frac{1}{4} x^2 e^{-2x} - \frac{1}{4} \int x\, (e^{-2x})'\, dx$$

$$\Rightarrow C_2(x) = -\frac{1}{4} x^2 e^{-2x} - \frac{1}{4} x e^{-2x} + \frac{1}{4} \int e^{-2x}\, dx$$

$$\Rightarrow C_2(x) = -\frac{1}{4} x^2 e^{-2x} - \frac{1}{4} x e^{-2x} - \frac{1}{8} e^{-2x}$$

$$C_1'(x) + \frac{1}{2} x^2 e^{-2x} e^{2x} = 0$$

$$\Rightarrow C_1'(x) + \frac{1}{2} x^2 = 0$$

$$\Rightarrow C_1'(x) = -\frac{1}{2} x^2$$

$$\Rightarrow C_1(x) = -\frac{1}{2} \int x^2\, dx$$

$$\Rightarrow C_1(x) = -\frac{1}{2} \frac{x^3}{3} = -\frac{x^3}{6}$$

Άρα

$$y_\mu(x) = -\frac{x^3}{3} + \left[-\frac{1}{4}x^2 e^{-2x} - \frac{1}{4}x e^{-2x} - \frac{1}{8}e^{-2x}\right] e^{2x}$$

$$= -\frac{x^3}{3} - \frac{1}{4}x^2 - \frac{1}{4}x - \frac{1}{8}$$

και επομένως

$$y(x) = c_1 + c_2 e^{2x} - \frac{x^3}{3} - \frac{1}{4}x^2 - \frac{1}{4}x - \frac{1}{8}$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

ΘΕΜΑ 6º

$$\vec{F} = \left(\sin y^2 + z^3,\ 2xy\cos y^2 - 2,\ 3xz^2 + 4\right)$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y^2 + z^3 & 2xy\cos y^2 - 2 & 3xz^2 + 4 \end{vmatrix} =$$

$$= \vec{i} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy\cos y^2 & 3xz^2 + 4 \end{vmatrix} - \vec{j} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ \sin y^2 + z^3 & 3xz^2 + \end{vmatrix}$$

$$+ \vec{k} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \sin y^2 + z^3 & 2xy\cos y^2 - 2 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(3xz^2+4) - \frac{\partial}{\partial z}(2xy\cos y^2)\right] -$$

$$-\vec{j}\left[\frac{\partial}{\partial x}(3xz^2+4) - \frac{\partial}{\partial z}(\sin y^2 + z^3)\right] +$$

$$+\vec{k}\left[\frac{\partial}{\partial x}(2xy\cos y^2) - \frac{\partial}{\partial y}(\sin y^2 + z^3)\right] =$$

$$= \vec{i}\,(0-0) - \vec{j}\,(3z^2 - 3z^2) +$$

$$+ \vec{k}\,(2y\cos y^2 - 2y\cos y^2)$$

$$= \vec{i}\cdot 0 + \vec{j}\cdot 0 + \vec{k}\cdot 0 = \vec{0}$$

Άρα η $\vec{F}$ είναι συντηρητικό πεδίο.

ΘΕΜΑ 7ο

$f = f(x,y) = x^3 + 6y^3 + 3x^2 + 9y^2$

$\nabla f = 0 \Rightarrow \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = 0 \Rightarrow$

$$\Rightarrow \begin{cases} \frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2 + 6x = 0 \\ \frac{\partial f}{\partial y} = 0 \Rightarrow 18y^2 + 18y = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 3x(x+2) = 0 \\ 18y(y+1) = 0 \end{cases} \Rightarrow \begin{cases} x = 0,\ x = -2 \\ y = 0,\ y = -1 \end{cases}$$

$A(0,0)$ $B(0,-1)$

$\Gamma(-2,0)$ $\Delta(-2,-1)$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

20/8/2001

ΘΕΜΑ 1ο

$$y'' - 2y' = 3e^{2x}x^2$$

$$y(x) = y_o(x) + y_\mu(x)$$

• Για την $y_o(x)$:

$$y'' - 2y' = 0$$

$$r^2 - 2r = 0$$

$$\Rightarrow r(r-2) = 0 < \begin{matrix} r_1 = 0 \\ r_2 = 2 \end{matrix}$$

$$y_o(x) = c_1 e^{2x} + c_2$$

• Για την $y_\mu(x)$

$$y_\mu(x) = c_1(x) e^{2x} + c_2(x)$$

$$\begin{cases} c_1'(x) e^{2x} + c_2'(x) = 0 \\ 2c_1'(x) e^{2x} + 0 = 3e^{2x} x^2 \end{cases}$$

$$2c_1'(x) \cancel{e^{2x}} = 3x^2 \cancel{e^{2x}}$$

$$\Rightarrow c_1'(x) = \frac{3}{2} x^2$$

$$\Rightarrow c_1(x) = \frac{1}{2} \int 3x^2 dx$$

$$\Rightarrow C_1(x) = \frac{1}{2} x^3$$

$$C_1'(x)\, e^{2x} + C_2'(x) = 0$$

$$\Rightarrow \frac{3}{2} x^2 e^{2x} + C_2'(x) = 0$$

$$\Rightarrow C_2'(x) = -\frac{3}{2} x^2 e^{2x}$$

$$\Rightarrow C_2(x) = -\frac{3}{2} \int x^2 e^{2x}\, dx$$

$$\Rightarrow C_2(x) = -\frac{3}{4} \int x^2 (e^{2x})'\, dx$$

$$\Rightarrow C_2(x) = -\frac{3}{4} x^2 e^{2x} + \frac{3}{\cancel{4}_2} \int \cancel{2}x\, e^{2x}\, dx$$

$$\Rightarrow C_2(x) = -\frac{3}{4} x^2 e^{2x} + \frac{3}{2} \int x\, e^{2x}\, dx$$

$$\Rightarrow C_2(x) = -\frac{3}{4} x^2 e^{2x} + \frac{3}{4} \int x\, (e^{2x})'\, dx$$

$$\Rightarrow C_2(x) = -\frac{3}{4} x^2 e^{2x} + \frac{3}{4} x e^{2x} - \frac{3}{4} \int e^{2x}\, dx$$

$$\Rightarrow C_2(x) = -\frac{3}{4} x^2 e^{2x} + \frac{3}{4} x e^{2x} - \frac{3}{8} e^{2x}$$

Άρα

$$y_p(x) = \frac{1}{2} x^3 e^{2x} - \frac{3}{4} x^2 e^{2x} + \frac{3}{4} x e^{2x} - \frac{3}{8} e^{2}$$

$$\Rightarrow y_p(x) = \left(\frac{x^3}{2} - \frac{3x^2}{4} + \frac{3x}{4} - \frac{3}{8} \right) e^{2x}$$

Sea

$$y(x) = C_1 e^{2x} + C_2 + \left(\frac{x^3}{2} - \frac{3x^2}{4} + \frac{3x}{4} - \frac{3}{8} \right) e^{2x}$$

ΘΕΜΑ 2º

1ος τρόπος

$$(x+y)dx + (x-y)dy = 0 \quad (*)$$

$P(x,y) = x+y$

$Q(x,y) = x-y$

$$\left.\begin{array}{l} \frac{\partial P}{\partial y} = 1 \\ \frac{\partial Q}{\partial x} = 1 \end{array}\right\} \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 1$$

Άρα η δοθείσα δ.ε. είναι ακριβής και:

$$\int_{x_0}^{x} P(t,y)\,dt + \int Q(x_0,y)\,dy = C$$

Παίρνω $x_0 = 0$, άρα:

$$\int_0^x P(t,y)\,dt + \int Q(0,y)\,dy = C$$

$$\Rightarrow \int_0^x (t+y)\,dt + \int -y\,dy = C$$

$$\Rightarrow \left.\frac{t^2}{2} + yt\right|_0^x - \frac{y^2}{2} = C$$

$$\Rightarrow \frac{x^2}{2} + yx - \frac{y^2}{2} = C$$

$$\Rightarrow x^2 + 2yx - y^2 = C_1 \quad (C_1 = 2C)$$

2ος τρόπος

$$(x+y)dx + (x-y)dy = 0$$

$$P(x,y) = x+y$$
$$Q(x,y) = x-y$$

$$P(\kappa x, \kappa y) = \kappa(x+y) = \kappa P(x,y)$$
$$Q(\kappa x, \kappa y) = \kappa(x-y) = \kappa Q(x,y)$$

Άρα η δοθείσα είναι ομογενής βαθμού ομογένειας 1.

$$\left.\begin{array}{l} (x+y)dx + (x-y)dy = 0 \\ \text{Θέτω } y = u \cdot x \\ dy = u\,dx + x\,du \end{array}\right\} \Rightarrow$$

$$\Rightarrow (x+ux)\,dx + (x-ux)(x\,du + u\,dx) = 0$$

$$\Rightarrow x(1+u)\,dx + x(1-u)\,x\,du + x(1-u)\,u\,dx$$

$$\Rightarrow \cancel{x}(1+u)\,dx + \cancel{x}u(1-u)\,dx + x^{\cancel{2}}(1-u)\,du =$$

$$\Rightarrow (1+u+u-u^2)\,dx + x(1-u)\,du = 0$$

$$\Rightarrow (1+2u-u^2)\,dx + x(1-u)\,du = 0$$

$$\Rightarrow \frac{1}{x}\,dx + \frac{1-u}{1+2u-u^2}\,du = 0$$

$$\Rightarrow \int \frac{1}{x}\,dx + \int \frac{1-u}{1+2u-u^2}\,du = c$$

$$\Rightarrow \ln x + \frac{1}{2}\int \frac{2-2u}{1+2u-u^2}\,du = c$$

$$\Rightarrow \ln x + \frac{1}{2}\ln(1+2u-u^2) = c$$

$$\Rightarrow 2\ln x + \ln(1+2u-u^2) = c_1 = 2c$$

$$\Rightarrow \ln x^2(1+2u-u^2) = c_1$$

$$\Rightarrow x^2(1+2u-u^2) = c_2 = e^{c_1}$$

$$\Rightarrow x^2\left(1+2\cdot\frac{y}{x} - \frac{y^2}{x^2}\right) = c_2$$

$$\Rightarrow x^2 + 2xy - y^2 = c_2$$

[illegible] $(n+1)$ [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

ΘΕΜΑ 3º

$$x^2y' - x^2y^2 - xy - 1 = 0$$

i) $y = -\frac{1}{x}$:

$$x^2\left(-\frac{1}{x}\right)' - x^2\left(-\frac{1}{x}\right)^2 - x\left(-\frac{1}{x}\right) - 1 = 0$$

$$\Rightarrow \cancel{x^2}\cdot\frac{1}{\cancel{x^2}} - \cancel{x^2}\cdot\frac{1}{\cancel{x^2}} + \cancel{x}\cdot\frac{1}{\cancel{x}} - 1 = 0$$

$$\Rightarrow 1 - 1 + 1 - 1 = 0$$

$\Rightarrow 0 = 0$ ισχύει.

ii)

$$x^2y' - x^2y^2 - xy - 1 = 0$$

$$\Rightarrow y' = y^2 + \frac{1}{x}y + \frac{1}{x^2}$$

Έχει ειδική λύση την $y_1 = -\frac{1}{x}$

Θέτω $y(x) = y_1(x) + \frac{1}{u(x)}$

$$\Rightarrow y = -\frac{1}{x} + \frac{1}{u}$$

και $y' = \frac{1}{x^2} - \frac{u'}{u^2}$ $\Rightarrow$

$$\Rightarrow y' = y^2 + \frac{1}{x} y + \frac{1}{x^2}$$

$$\Rightarrow \cancel{\frac{1}{x^2}} - \frac{u'}{u^2} = \left(-\frac{1}{x} + \frac{1}{u}\right)^2 + \frac{1}{x}\left(-\frac{1}{x} + \frac{1}{u}\right) + \cancel{\frac{1}{x^2}}$$

$$\Rightarrow -\frac{u'}{u^2} = \cancel{\frac{1}{x^2}} + \frac{1}{u^2} - \frac{2}{xu} - \cancel{\frac{1}{x^2}} + \frac{1}{xu}$$

$$\Rightarrow -\frac{u'}{u^2} = \frac{1}{u^2} - \frac{1}{xu}$$

$$\Rightarrow \frac{u'}{u^2} = -\frac{1}{u^2} + \frac{1}{xu}$$

$$\Rightarrow xu' = -x + u$$

$$\Rightarrow u' \underbrace{- \frac{1}{x}}_{p(x)} u + \underbrace{1}_{q(x)} = 0 \rightarrow \text{γραμμική}$$

$$\mu(x) = e^{\int p(x)dx} = e^{-\int \frac{1}{x}dx} = e^{-\ln x} = x^{-1} = \frac{1}{x}$$

Άρα $u(x) = e^{-\int p(x)dx}\left[c + \int q(x) e^{\int p(x)dx} dx\right]$

$$\Rightarrow u(x) = x\left[c + \int \frac{1}{x} dx\right]$$

$$\Rightarrow u(x) = x\left[c + \ln x\right]$$

Άρα $y(x) = -\frac{1}{x} + \frac{1}{x[c+\ln x]}$

16/6/2000

Θέμα 1ο

$$(4xy+3y^4)dx+(2x^2+5xy^3)dy=0$$

Θεωρώ πολ/στη Euler της μορφής
$\mu(x,y)=x^a y^b$

Τότε η δ.ε:

$$\mu(x,y)(4xy+3y^4)dx+\mu(x,y)(2x^2+5xy^3)dy=0$$

είναι ακριβής

$$\Rightarrow x^a y^b(4xy+3y^4)dx+x^a y^b(2x^2+5xy^3)dy=0$$

$$\Rightarrow \underbrace{(4x^{a+1}y^{b+1}+3x^a y^{b+4})}_{P(x,y)}dx+\underbrace{(2x^{a+2}y^b+5x^{a+1}y^{b+3})}_{Q(x,y)}dy=0$$

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$$

$$\Rightarrow \frac{\partial}{\partial y}(4x^{a+1}y^{b+1}+3x^a y^{b+4})=\frac{\partial}{\partial x}(2x^{a+2}y^b+5x^{a+1}y^{b+3})$$

$$\Rightarrow 4(b+1)x^{a+1}y^b+3(b+4)x^a y^{b+3}=$$

$$=2(a+2)x^{a+1}y^b+5(a+1)x^a y^{b+3}$$

$$\Rightarrow \begin{cases} 4(b+1) = 2(a+2) \\ 3(b+4) = 5(a+1) \end{cases}$$

$$\Rightarrow \begin{cases} 2(b+1) = a+2 \\ 3b+12 = 5a+5 \end{cases}$$

$$\Rightarrow \begin{cases} 2b + \cancel{2} = a + \cancel{2} \\ 3b+12 = 5a+5 \end{cases}$$

$$\Rightarrow \begin{cases} 2b = a \\ 3b - 5a = 5-12 \end{cases}$$

$$\Rightarrow 3b - 5 \cdot 2b = -7$$

$$3b - 10b = -7$$

$$-7b = -7$$

$$b = 1.$$

Άρα $a = 2$

Άρα ο πολλ/στης Euler είναι ο

$\mu(x,y) = x^2 y.$

$$(4xy+3y^4)\,dx+(2x^2+5xy^3)\,dy=0$$

$$\to x^2y\,(4xy+3y^4)\,dx+x^2y\,(2x^2+5xy^3)\,dy=0$$

ακριβής

$$\Rightarrow (4x^3y^2+3y^{\cancel{5}}x^2)\,dx+(2x^4y+5x^3y^4)\,dy=0$$

$$\Rightarrow \int_0^x (4t^3y^2+3y^4)\,dt+\int \cancel{0\cdot dy}=C$$

$$\Rightarrow t^4y^2+3y^4t\Big|_0^x=C$$

$$\Rightarrow x^4y^2+3y^4x=C$$

$$\Rightarrow xy^2(x^3+3y^2)=C$$

$$\int_{x_0}^{x} P(t,y)\,dt+\int Q(x_0,y)\,dy=C$$

$x_0=0$

$$C=\int(4x^3y^2+3y^4)\,dx+\int(2x^4y+5x^3y^4)\,dy-\int\left[\frac{\partial\, y^2x^4+3y^4x}{\partial y}\right]dy$$

$$x^4y^2+3y^4x+y^2x^4+x^3y^5-\int(2yx^4+12y^3x)\,dy$$

$$2x^4y^2+\cancel{3y^4x}+x^3y^5-(y^2x^4+\cancel{3y^4x})=0$$

$$x^4y^2+x^3y^5$$

ΘΕΜΑ 2°

$$y'' - 2y' + y = x^2 e^{2x}$$

$\hookrightarrow y(x) = y_0(x) + y_\mu(x)$

· Για τη $y_0(x)$:

$$y'' - 2y' + y = 0$$

$$r^2 - 2r + 1 = 0$$

$\Rightarrow (r-1)^2 = 0 \Rightarrow$

$\Rightarrow r = 1$ διπλή ρίζα

Άρα $y_0(x) = c_1 e^x + c_2 x e^x$

· Για τη $y_\mu(x)$:

$$y_\mu(x) = c_1(x) e^x + c_2(x) \cdot x e^x$$

$$\begin{cases} c_1'(x) e^x + c_2'(x) x e^x = 0 \\ c_1'(x) e^x + c_2'(x) (e^x + x e^x) = x^2 e^{2x} \end{cases}$$

$$\Rightarrow \begin{cases} c_1'(x) + c_2'(x) \cdot x = 0 \\ c_1'(x) + c_2'(x)(1+x) = x^2 e^x \end{cases}$$

Αφαιρώ κατά μέλη και έχω:

$\longrightarrow$

$$C_2'(x)\cdot x - C_2'(x)(1+x) = -x^2 e^x$$
$$\Rightarrow C_2'(x)\left[\cancel{x} - 1 - \cancel{x}\right] = -x^2 e^x$$

$$\Rightarrow -C_2'(x) = -x^2 e^x$$
$$\Rightarrow C_2'(x) = x^2 e^x$$

$$\Rightarrow C_2(x) = \int x^2 e^x \, dx$$

$$\Rightarrow C_2(x) = x^2 e^x - \int 2x e^x \, dx \Rightarrow$$
$$\Rightarrow C_2(x) = x^2 e^x - 2x e^x + 2\int e^x \, dx$$
$$\Rightarrow C_2(x) = x^2 e^x - 2x e^x + 2e^x$$

$$C_1'(x) + x^2 e^x \cdot x = 0$$
$$\Rightarrow C_1'(x) = -x^3 e^x$$
$$\Rightarrow C_1(x) = -\int x^3 e^x \, dx$$

$$\Rightarrow C_1(x) = -x^3 e^x + \int 3x^2 e^x \, dx$$
$$\Rightarrow C_1(x) = -x^3 e^x + 3x^2 e^x - 3\int 2x e^x \, dx$$
$$\Rightarrow C_1(x) = -x^3 e^x + 3x^2 e^x - 6x e^x + 6\int e^x \, dx$$
$$\Rightarrow C_1(x) = -x^3 e^x + 3x^2 e^x - 6x e^x + 6e^x$$

$$y_r(x) = (x^2 - 2x + 2)e^{2x} + (-x^3 + 3x^2 - 6x + 6)\, x\, e^{2x}$$
$$= (x^2 - 2x + 2)e^{2x} + (-x^4 + 3x^3 - 6x^2 + 6x)e^{2x}$$
$$= (x^2 - 2x + 2 - x^4 + 3x^3 - 6x^2 + 6x)e^{2x}$$
$$= (-x^4 + 3x^3 - 8x^2 + 4x + 2)e^{2x}$$

Άρα $y(x) = (C_1 + C_2 x)e^x +$
$$+ (-x^4 + 3x^3 - 8x^2 + 4x + 2)e^{2x}$$

ΘΕΜΑ 3º

$$y' = 2xy(x^2y^2-1)$$

$$\Rightarrow y' = 2x^3y^3 - 2xy$$

$$\Rightarrow y' + 2xy = 2x^3y^3$$

$$\Rightarrow y^{-3}y' + 2xy^{-2} = 2x^3$$ Bernoulli

Θέτω $z = y^{-2}$

$$z' = -2y^{-3}y'$$

$$\Rightarrow -\frac{1}{2}z' = y^{-3}y'$$

$$\Rightarrow -\frac{1}{2}z' + 2xz = 2x^3$$

$$\Rightarrow z' \underbrace{-4x}_{P(x)} z = \underbrace{-4x^3}_{q(x)}$$ γραμμική

$$\mu(x) = e^{\int P(x)dx} = e^{-\int 4x dx} = e^{-2x^2}$$

Άρα

$$z(x) = e^{2x^2}\left[c + \int -4x^3 \cdot e^{-2x^2}dx\right] \quad (*)$$

$$\int -4x^3e^{-2x^2}dx = \int x^2(-4xe^{-2x^2})dx =$$

$$= \int x^2(e^{-2x^2})'dx = x^2e^{-2x^2} - 2\int xe^{-2x^2}d$$

$$= x^2e^{-2x^2} + \frac{1}{2}\int -4xe^{-2x^2}dx =$$

$$= x^2 e^{-2x^2} + \frac{1}{4}\int \left(e^{-2x^2}\right)' dx =$$

$$= x^2 e^{-2x^2} + \frac{1}{4} e^{-2x^2} = \left(x^2 + \frac{1}{4}\right) e^{-2x^2}$$

$$(*) : \quad z(x) = e^{2x^2}\left[c + \left(x^2 + \frac{1}{4}\right) e^{-2x^2}\right]$$

ΘΕΜΑ 4º

$$y'' - 2y' = 12x^2 + \sin 2x$$

$$y(x) = y_0(x) + y_\mu(x)$$

• Για την $y_0(x)$:

$$y'' - 2y' = 0$$

$$r^2 - 2r = 0$$

$$\Rightarrow r(r-2) = 0 < \begin{matrix} r_1 = 0 \\ r_2 = 2 \end{matrix}$$

$$y_0(x) = c_1 + c_2 e^{2x}$$

• Για τη $y_\mu(x)$:

$$y_\mu(x) = y_1(x) + y_2(x)$$

$y_1(x)$ λύση της $y'' - 2y' = 12x^2$ (*)
$y_2(x)$ λύση της $y'' - 2y' = \sin 2x$ (**)

i) Για τη $y_1(x)$:

$$y'' - 2y' = 12x^2$$

$$12x^2 = e^{0x}\left[12x^2 \cos 0x + 0 \cdot \sin 0x\right]$$

$\left.\begin{matrix} A = 0 \\ B = 0 \end{matrix}\right\}$ $A + Bi = \underline{0}$ είναι ρίζα του χαρακτ. πολυωνύμου πολ/τας $\underline{\lambda = 1}$

πολυώνυμο 2ου βαθμού

$$y_1(x) = e^{0x}\cdot x^1\left[\boxed{Q_1(x)\cos 0x + \cancel{Q_2(x)\sin 0x}}\right]$$

$$\Rightarrow y_1(x) = x\cdot\left[\alpha x^2+\beta x+\gamma\right]$$

$$\Rightarrow y_1(x) = \alpha x^3+\beta x^2+\gamma x$$

$$\left.\begin{aligned} y_1'(x) &= 3\alpha x^2+2\beta x+\gamma \\ y_1''(x) &= 6\alpha x+2\beta \end{aligned}\right\}$$

Αντικαθιστώ στην (*)

$$6\alpha x+2\beta-2(3\alpha x^2+2\beta x+\gamma)=12x^2$$

$$\Rightarrow 6\alpha x+2\beta-6\alpha x^2-4\beta x-2\gamma=12x^2$$

$$\Rightarrow -6\alpha x^2+(6\alpha-4\beta)x+2\beta-2\gamma=12x^2$$

$$\Rightarrow \begin{cases} -6\alpha=12 \\ 6\alpha-4\beta=0 \\ 2\beta-2\gamma=0 \end{cases} \Rightarrow \begin{cases} \alpha=-2 \\ 3\alpha-2\beta=0 \\ \beta-\gamma=0 \end{cases}$$

$$\Rightarrow \begin{cases} \alpha=-2 \\ \beta=-3 \\ \gamma=-3 \end{cases}$$

Άρα $\underline{y_1(x) = -2x^3-3x^2-3}$

ii) Για την $y_2(x)$:

$$y'' - 2y' = \sin 2x$$

$$\sin 2x = e^{0x}\left[0\cdot\cos 2x + 1\cdot\sin 2x\right]$$

$A = 0$, $B = 2$ $\Rightarrow A+Bi = 0+2i = 2i$ δεν είναι ρίζα του χαρακτηριστ. πολυωνύμου άρα $\lambda = 0$

$$y_2(x) = e^{0x}x^0\left[C_3\cdot\cos 2x + C_4\cdot\sin 2x\right]$$

$$\Rightarrow \left.\begin{aligned} y_2(x) &= C_3\cos 2x + C_4\sin 2x \\ y_2'(x) &= -2C_3\sin 2x + 2C_4\cos 2x \\ y_2''(x) &= -4C_3\cos 2x - 4C_4\sin 2x \end{aligned}\right\}$$

Αντικαθιστώ στην (**)

$$-4C_3\cos 2x - 4C_4\sin 2x - 2(-2C_3\sin 2x + 2C_4\cos 2x) = \sin 2x$$

$$\Rightarrow -4C_3\cos 2x - 4C_4\sin 2x + 4C_3\sin 2x - 4C_4\cos 2x = \sin 2x$$

$$\Rightarrow -4(C_3+C_4)\cos 2x - 4(C_4-C_3)\sin 2x = \sin 2x$$

$$\Rightarrow \begin{cases} \cancel{-4}(C_3+C_4)=0 \\ \cancel{-4}(C_4-C_3)=0 \end{cases}$$

$$\Rightarrow \begin{cases} C_3+C_4=0 \\ C_4=C_3 \end{cases}$$

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ΘΕΜΑ 1ο

$$(3x-3y-1)\,dx + (x-y+3)\,dy = 0$$

$$\Rightarrow (3x-3y-1)\,dx = -(x-y+3)\,dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x-3y-1}{-(x-y+3)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{3(x-y)-1}{-(x-y)-3}$$

Θέτω $z = x-y$

$$\frac{dz}{dx} = 1 - \frac{dy}{dx}$$

$$\Rightarrow \frac{dz}{dx} = 1 - \frac{3z-1}{-z-3}$$

$$\Rightarrow \frac{dz}{dx} = 1 + \frac{3z-1}{z+3}$$

$$\Rightarrow \frac{dz}{dx} = \frac{z+3+3z-1}{z+3}$$

$$\Rightarrow \frac{dz}{dx} = \frac{4z+2}{z+3}$$

$$\Rightarrow \frac{z+3}{4z+2}\,dz = dx$$

$$\Rightarrow \int \frac{z+3}{4z+2}\,dz = \int dx + C$$

[illegible]

[illegible]

[illegible]